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Comparaison entre Taylor et méthode de Simpson modifiée
pour un problème initiale

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:

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Introduction

Many problems of mathematical physics can be started in the form of initial and boundary value problems associated with the ordinary and partial differential equations. These equations also occur as reformulations of other mathematical problems such as integral equations can be cast into the Volterra integral equations.

This introduction contains a brief description of the objective, approach, and organization of the thesis. The objective of this work is to explain the most numerical methods, For example Taylor, Rang kutta and Euler which can be used to solve linear differential equations with the coefficient is functions continuous of the first kind that cannot be solved analytically. The most general form of a linear differential equation is

$$\begin{aligned}y' + a(x)y &= f(x) \\ y(x_0) &= y_0\end{aligned}$$

for we obtain results the most approximate to exact solution we transform differential equation to volterra integral equation, It becomes on form

$$\varphi(x) = g(x) + \int_a^x k(x, t)\varphi(t)dt \quad a \leq x \leq b$$

where the function $g(x)$ and the regular kernel $k(x, t)$ are given, and $\varphi(x)$ is the unknown function to be determined.

In general, it is not possible to solve an integral equation analytically, Therefore we turning solve it by the numerical methods For example, repeated trapezoidal rule, Newton-Cotes, Clenshaw-Curtis, and Simpson's rule. This is, of course, not the only way to approximate Volterra integral equations.

In this work we study the numerical solutions of the differential equation and some linear integral equations with continuous kernel. This thesis consists of introduction, three chapters

In The first Chapter will present few basic concept from general about Operator Compact, such as compactness, The linear combination and product between two Operator Compact, and we'll talk in this chapter about theorem Weakly Singular Kernel

In The second Chapter we will mention Existence and Uniqueness theorem of the differential equation first and second order, And we try to apply the numerical method of Taylor for solving differential equation, for approximating to exact solution by programming, we used the preferred language in numerical analysis in Matlab. The chapter contains also how transform differential equation two first order to Integral equation.

In The last Chapter we apply the concept of Volterra Integral equation, and how modified method of Simpson and we try solving Volterra integral equation two first order by modified simpson method for obtaining approximate solution of the linear integral equation

The main goal of this is thesis to comparison the results between solution of differential equation by Taylor method and solution of Volterra integral equation by modified simpson method

Chapter 1

Operator Compact

1.1 Compact Linear operators

A linear operator A defined from a normed space E into normed space F is called a linear compact operator or completely continuous linear operator if for every bounded subset G of E , the image $A(G)$ is relatively compact in F . In other words, the closure $\overline{A(G)}$ is compact.

Theorem 1.1.1 (*Compactness criterion*)

A linear operator A defined from a normed space E into a normed space F is called a linear compact operator or completely continuous linear operator if and only if for every bounded sequence φ_n in E , the sequence $A\varphi_n$ in F has a convergence.

Let φ_n be a bounded sequence in E , since the operator A is compact, then the set $\{A\varphi_n\}$ is relatively compact in F where this property shows that $A\varphi_n$ contains a convergent subsequence.

Conversely, let us consider any bounded subset G in E and let ψ_n be any sequence in $A(G)$. Then there exists a bounded sequence φ_n in G , such that

$$\psi_n = A\varphi_n.$$

By assumption, $A\varphi_n = \psi_n$ contains a convergent subsequence ψ_{n_k} in F . Thus $A(G)$ is relatively compact, because for any bounded sequence ψ_n in $A(G)$ there exists a convergent

subsequence ψ_{n_k} in F . In other words, for all bounded set $G \subset E$, the set $A(G)$ is relatively compact in F . Hence A is compact.

The linear combination $A = \alpha A_1 + \beta A_2$ of compact operators A_1 and A_2 is a compact operator, for every scalars α and β .

Let φ_n be a bounded sequence in E and let $A\varphi_n$ be a sequence in F , then

$$A\varphi_n(x) = \alpha A_1\varphi_n(x) + \beta A_2\varphi_n(x), \quad \text{with } \varphi_n \in E, \quad n \in \mathbb{N}.$$

The operators A_1 and A_2 are compact, one can extract from $A_1\varphi_n$ and $A_2\varphi_n$ two convergent subsequences which give by their sum a convergent subsequence of $A\varphi_n$. Hence A is compact.

The product AB of two bounded operators A and B is compact if either of operators A or B is compact.

Let φ_n be a bounded sequence in E , then if we consider $B\varphi_n$ as a bounded, and from the compactness of the operator A gives a convergent subsequence $A(B\varphi_n(x))$ of $A(B\varphi_n(x))$. Hence the operator AB is compact

On the other hand, if we consider B as a compact, one can extract from $B\varphi_n(x)$ a convergent subsequence $B\varphi_{n(k)}(x)$, and from boundedness of the operator A gives the convergence of the sequence $A(B\varphi_{n(k)}(x))$. Hence the operator AB is compact.

A sequence A_n of compact operators defined from a normed space E into a Banach space F converges uniformly to an operator A , say,

$$\lim_{n \rightarrow \infty} \|A_n - A\| = 0$$

Then the limit operator A is compact.

Let φ_n be bounded sequence in E , the operator A_1 is compact, then one can extract from the sequence $A_1\varphi_n$ a convergent subsequence, say φ_n^1 a subsequence from φ_n such that $A_1\varphi_n^1$ converges.

In the same way, we can extract from the sequence $A_2\varphi_n^1$ a convergent subsequence, say φ_n^2 a subsequence from φ_n^1 such that $A_2\varphi_n^2$ converges.

Noting that, we obtain from the bounded sequence φ_n a subsequence φ_n^2 such that $A_1\varphi_n^2$ and $A_2\varphi_n^2$ both converge.

Continuing in this way, we see that, for the compact operators A_1, A_2, \dots, A_p there exists a nested subsequences

$$\varphi_n^p \subset \dots \varphi_n^2 \subset \varphi_n^1 \subset \varphi_n$$

such that, the sequences $A_k \varphi_n^p$ converge for all $k = 1, 2, \dots, p$.

In order to show the compactness of the operator limit A , we must use the completeness of the space F and showing that the sequence $A \varphi_n^p$ is Cauchy sequence.

Noting that the sequence φ_n is bounded, say $\|\varphi_n\| \leq M$ for all n . Hence $\|\varphi_n^p\| \leq M$ for each n and p . Choose $n = p$ so that

$$\|A_n - A\| < \frac{\varepsilon}{3M}.$$

Since the sequence $A_n \varphi_n^p$ is Cauchy, because it converges there is N such that, for all $p > N$ and $q > N$, we get

$$\|A_n \varphi_n^p - A_n \varphi_n^q\| < \frac{\varepsilon}{3}.$$

Hence we obtain

$$\begin{aligned} \|A \varphi_n^p - A \varphi_n^q\| &= \|A \varphi_n^p - A_n \varphi_n^p + A_n \varphi_n^p - A_n \varphi_n^q + A_n \varphi_n^q - A \varphi_n^q\| \\ &\leq \|A \varphi_n^p - A_n \varphi_n^p\| + \|A_n \varphi_n^p - A_n \varphi_n^q\| + \|A_n \varphi_n^q - A \varphi_n^q\| \\ &\leq \|A_n - A\| \|\varphi_n^p\| + \|A_n \varphi_n^p - A_n \varphi_n^q\| + \|A_n - A\| \|\varphi_n^q\| \\ &\leq \frac{\varepsilon}{3M} M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M} M = \varepsilon. \end{aligned}$$

Remembering that, due to the completeness of the space F , the Cauchy sequence $A \varphi_n^p$ converges as a subsequence of $A \varphi_n$ where φ_n^p is a subsequence of an arbitrary bounded sequence φ_n . Hence the compactness of the operator A .

Theorem 1.1.2 (*finite dimensional range*)

Let A be a bounded operator defined from E into F with the range $A(E)$ has a finite dimension $\dim A(E) < \infty$, then the operator A is compact.

Indeed, for all bounded set G in E , the range $A(G)$ is a bounded set in the finite dimensional space $A(E)$. Hence $A(G)$ is relatively compact, it follows that A is a compact operator.

Theorem 1.1.3 (*finite dimensional domain*)

Let A be a bounded operator defined from E into F with the domain E has a finite dimension $\dim E < \infty$, then the operator A is compact.

Indeed, the space E has a finite dimension $\dim E < \infty$ implies the finite dimensional range $A(E)$, say

$$\dim A(E) \leq \dim E$$

it follows that A is compact operator.

Let G be a closed subspace in the normed space E such that, $G \neq E$ then there exists an element $\varphi \in E$ with $\|\varphi\| = 1$ such that, for all $\psi \in G$, we have

$$\|\varphi - \psi\| \geq \alpha, \text{ with } 0 < \alpha < 1$$

Indeed, let f be an element of E such that $f \notin G$ then, we get

$$\inf_{h \in G} \|f - h\| = \beta > 0,$$

choosing an element g belongs to G such that,

$$\beta \leq \|f - g\| \leq \frac{\beta}{\alpha}.$$

Define the vector φ by

$$\varphi = \frac{f - g}{\|f - g\|},$$

this vector φ has a unit norm $\|\varphi\| = 1$, besides, for all $\psi \in G$ we get

$$\begin{aligned} \|\varphi - \psi\| &= \left\| \frac{f - g}{\|f - g\|} - \psi \right\| \\ &= \frac{1}{\|f - g\|} \|f - [g + (\|f - g\| \psi)]\| \\ &\geq \frac{\beta}{\|f - g\|} \geq \alpha. \end{aligned}$$

The identity operator I defined from a normed space E into E is compact if and only if the space E has a finite dimension let φ_1 be an element of E , such that $\|\varphi_1\| = 1$, then the

set of finite dimension $G_1 = \text{span} \{\varphi_1\}$ represents a closed subspace of E . So there exists an element $\varphi_2 \in E$, such that $\|\varphi_2\| = 1$ and $\|\varphi_1 - \varphi_2\| \geq \frac{1}{2}$. By the same way we take a closed subspace $G_2 = \text{span} \{\varphi_1, \varphi_2\}$ and finding an element $\varphi_3 \in E$ such that $\|\varphi_2\| = 1$ with $\|\varphi_2 - \varphi_3\| \geq \frac{1}{2}$. One repeat the same procedure until the obtaining of sequence φ_n verifying $\|\varphi_n\| = 1$ and $\|\varphi_m - \varphi_n\| \geq \frac{1}{2}$, for all $m \neq n$.

Noting that, the sequence φ_n is bounded but does not contain any convergent subsequence. Hence the operator $I\varphi_n = \varphi_n$ is not compact.

The closed unit ball $B(0, 1)$ in the normed space E of infinitely dimensional is not compact.

Indeed, $B(0, 1)$ is bounded but cannot be compact; thus

$$I \left(B(0, 1) = B(0, 1) = \overline{B(0, 1)} \right),$$

is not relatively compact.

A bounded operator A in a normed space E is not generally a compact operator.

Indeed, see the Identity operator $A = I$ in the infinitely dimensional normed space E .

The integral operator A defined from $C(G)$ into $C(G)$

$$A\varphi(x) = \int_x k(x, y) \varphi(y) dy, \quad x, y \in G$$

with continuous kernel $k(x, y)$ is a compact operator.

Let E be a bounded set of $C(G)$ then, for each $\varphi \in E$, we have

$$\|\varphi\| \leq M,$$

besides, for all $x \in G$ and $\varphi \in E$, we get

$$\begin{aligned} |A\varphi(x)| &= \left| \int_G k(x, y) \varphi(y) dy \right| \\ &\leq M |G| \max_{x, y \in G} |k(x, y)|. \end{aligned}$$

It follows that $A(E)$ is bounded.

By assumption, the kernel $k(x, y)$ is continuous over the compact $G \times G$, thus it is uniformly continuous and therefore

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, y, z \in G, |x - y| < \delta \implies |k(x, z) - k(y, z)| < \frac{\varepsilon}{M|G|}.$$

Hence, for each $\varphi \in E$ and $x, y \in G$, with $|x - y| < \delta$

$$\begin{aligned} |A\varphi(x) - A\varphi(y)| &= \left| \int_G (k(x, z) - k(y, z)) \varphi(z) dz \right| \\ &< \frac{\varepsilon}{M|G|} M|G| = \varepsilon \end{aligned}$$

This relation expresses that $A(E)$ is equicontinuous. Hence $A(E)$ is relatively compact, so by Arzela-Ascoli's theorem A is compact.

Weakly singular kernel

The kernel $k(x, y)$ is said to be weakly singular if it is defined continuous on $G \times G \subset \mathbb{R}^n \times \mathbb{R}^n$ for all $x \neq y$ and there exist a positive constants M and $\alpha \in]0, n]$ such that

$$|k(x, y)| < \frac{M}{|x - y|^{n-\alpha}}, \quad x, y \in G, \quad x \neq y.$$

In other words,

$$\forall x, y \in G, \quad x \neq y, \quad \exists M > 0, \quad |k(x, y)| < \frac{M}{|x - y|^{n-\alpha}}, \quad 0 < \alpha < n$$

The integral operator A defined from $C(G)$ into $C(G)$ with weakly continuous kernel is compact operator.

Noting that, the integral operator

$$A\varphi(x) = \int_G k(x, y) \varphi(y) dy, \quad x, y \in G$$

exists as an improper integral, due to the weakly continuous kernel

$$|k(x, y)| \varphi(y) \leq M \|\varphi\| |x - y|^{n-\alpha},$$

further,

$$\int_G |x - y|^{n-\alpha} dy \leq w_n \int_0^d \rho^{\alpha-n} \rho^{n-1} d\rho = \frac{w_n}{\alpha} d^\alpha,$$

where w_n designates the surface area of the unit sphere in \mathbb{R}^n and d the diameter of the set G .

Let us construct a sequence of compact operators A_p which converges to the integral operator A , such that

$$\lim_{n \rightarrow \infty} \|A_p - A\| = 0.$$

choosing now a linear continuous function h defined on $[0, \infty[$ into \mathbb{R} , by

$$h(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2} \\ 2t - 1 & \text{if } \frac{1}{2} \leq t \leq 1 \\ 1 & \text{if } 1 \leq t < \infty \end{cases}$$

The function $k_p(x, y)$ defined on $G \times G$ into \mathbb{R} , by

$$k_p(x, y) = \begin{cases} h(p|x-y|) k(x, y) & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

is a continuous kernel for each $p \in \mathbb{N}$. Hence the integral operators A_p such that

$$A_p \varphi(x) = \int_G k_p(x, y) \varphi(y) dy, \quad x, y \in G.$$

are compact.

Besides, for all $x \in G$, we get

$$\begin{aligned} |A_p \varphi(x) - A \varphi(x)| &= \left| \int_G [k_p(x, y) - k(x, y)] \varphi(y) dy \right| \\ &= \left| \int_{G \cap |x-y| < \frac{1}{p}} \{h(p|x-y|) - 1\} k(x, y) \varphi(y) dy \right| \\ &\leq M \|\varphi\| w_n \int_0^{\frac{1}{p}} \rho^{\alpha-n} \rho^{n-1} d\rho \\ &\leq M \|\varphi\| \frac{w_n}{\alpha \rho^\alpha}. \end{aligned}$$

It is simple to see that the convergence $A_p \varphi$ to $A \varphi$ is uniform, so it follows that,

$$\|A - A_p\| \leq M \frac{w_n}{\alpha \rho^\alpha} \rightarrow 0, \quad \text{when } p \rightarrow \infty,$$

and thus A is compact operator.

The integral operator A defined from the normed space $C(\partial G)$ into $C(\partial G)$ with continuous or weakly continuous kernel is a compact operator, where under ∂G we designate a regular boundary of the set G .

Chapter 2

The Resolution of the Initial Value Problems

2.1 Existence and Uniqueness:

2.1.1 First-Order Linear Differential Equations

is an equation of the form

$$a_1(x)y'(x) + a_0(x)y(x) = g(x) \quad (2.1.1)$$

where $a_0(x)$, $a_1(x)$ and $g(x)$ are given continuous functions of x on some interval. Values of x for which $a_1(x) = 0$ are called singular points of the differential equation. Equation (2.1.1) is said to be homogeneous if $g(x) = 0$; otherwise it is nonhomogeneous

A homogeneous linear differential equation $a_1y'(x) + a_0y(x) = 0$ is separable, so it can be solved implicitly. In an interval where $a_1(x) \neq 0$, we divide both sides of equation (2.1.1) by the leading coefficient $a_1(x)$ to reduce it to a more useful form, usually called the standard form of a linear equation:

$$y' + a(x)y = f(x) \quad (2.1.2)$$

where $a(x) = a_0(x)/a_1(x)$ and $f(x) = g(x)/a_1(x)$ are given continuous functions of x in some interval the initial value problem for equation (2.1.2) always has a unique solution

in the interval where both functions $a(x)$ and $f(x)$ are continuous. Moreover, a linear differential equation has no singular solution.

The solutions to equation(2.1.2) have the property that they can be written as the sum of two functions $y = y_h + y_p$,

where y_h is the general solution of the associated homogeneous equation $y' + a(x)y = 0$ and y_p is a particular solution of the nonhomogeneous equation $y' + a(x)y = f(x)$. The function y_h is usually referred to as the complementary function to equation (2.1.2).

In engineering applications, the independent variable often represents time and conventionally is denoted by t .

2.1.2 Second Order Linear Differential Equations:

A second order differential equation in the normal form is as follows:

$$y'' = F(x, y, y') \quad (2.1.3)$$

For given two numbers y_0 and y_1 , we impose the initial conditions on $y(x)$ in the form

$$y(x_0) = y_0, \quad y'(x_0) = y_1 \quad (2.1.4)$$

The differential equation (2.1.3) with the initial conditions (2.1.4) is called the initial value problem (**IVP** for short) or the Cauchy problem. Now we can state the following theorem, which is a direct extension of Picard's

The general linear differential equation of the second order is an equation that can be written as

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x). \quad (2.1.5)$$

Let $p(x)$, $q(x)$, and $f(x)$ be continuous functions on an open interval . Then, for each $x_0 \in (a, b)$, the initial value problem

$$y'' + p(x)y' + q(x)y = f(x), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

has a unique solution for arbitrary specified real numbers y_0, y_1 .

The existence and uniqueness:

An arbitrary differential equation of the first order $y' = f(x, y)$ does not necessarily have a solution that satisfies it. Therefore, the existence of a solution is an important problem for both the theory of differential equations and their applications.

Theorem 2.1.1 *Let us consider the initial value problem for the linear differential equation*

$$y' + a(x)y = f(x) \quad (2.1.6)$$

$$y(x_0) = y_0 \quad (2.1.7)$$

where $a(x)$ and $f(x)$ are known functions and y_0 is an arbitrary prescribed initial value. Assume that the functions $a(x)$ and $f(x)$ are continuous on an open interval $\alpha < x < \beta$ containing the point x_0 . Then the initial value problem (2.1.6), (2.1.7) has a unique solution $y = \varphi(x)$ on the same interval (α, β) .

proof. we show that if equation (2.1.6) has a solution, then it must be given by the following formula: ■

$$y(x) = \mu^{-1}(x) \left[\int \mu(x)f(x)dx + C \right], \quad \mu(x) = \exp \left\{ \int a(x)dx \right\} \quad (2.1.8)$$

When $\mu(x)$ is a non zero differentiable function on the interval (α, β) , we have from equation (2.1.6), that

$$\frac{d}{dx} [\mu(x)y(x)] = \mu(x)f(x).$$

Since both $\mu(x)$ and $f(x)$ are continuous functions, its product $\mu(x)f(x)$ is integrable, and formula (2.1.8) follows from the latter. Hence, from equation (2.1.8), the function $y(x)$ exists and is differentiable over the interval (α, β) . By substituting the expression for $y(x)$ into equation (2.1.6), one can verify that this expression is a solution of equation (2.1.6).

Finally, the initial condition (2.1.7) determines the constant C uniquely. If we choose the lower limit to be x_0 in all integrals in the expression (2.1.8), then

$$y(x) = \frac{1}{\mu(x)} \left[\int_{x_0}^x \mu(s) f(s) ds + y_0 \right], \quad \mu(x) = \exp \left\{ \int_{x_0}^x a(s) ds \right\}$$

is the solution of the initial value problem (2.1.7),

Theorem 2.1.2 (*Peano*):

Suppose that the function $f(x, y)$ is continuous in some rectangle:

$$\Omega = \{(x, y) : x_0 - a \leq x \leq x_0 + a, y_0 - b \leq y \leq y_0 + b\} \quad (2.1.9)$$

Let

$$M = \max_{(x,y) \in \Omega} |f(x, y)|, \quad (2.1.10)$$

Then the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \quad (2.1.11)$$

has a solution in the interval $[x_0 - h, x_0 + h]$.

Theorem 2.1.3 *Let $f(x, y)$ be a continuous function in a rectangular domain containing the point (x_0, y_0) . If $f(x, y)$ satisfies the Lipschitz condition*

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$$

for some positive constant L (called the Lipschitz constant) and any x, y_1 , and y_2 from then the initial value problem (2.1.11) has a unique solution in some interval $x_0 - h \leq x \leq x_0 + h$, where h is defined in equation (2.1.10).

We cannot guarantee that the solution $y = \varphi(x)$ of the initial value problem (2.1.11) exists in the interval $(x_0 - a, x_0 + a)$ because the integral curve $y = \varphi(x)$ can exist outside of the rectangle. For example, if there exists

x_1 such that $x_0 - a < x_1 < x_0 + a$ and $y_0 + b = \varphi(x_1)$, then for $x > x_1$ (if $x_1 > x_0$) the solution $\varphi(x)$ cannot be defined. We definitely know that the solution $y = \varphi(x)$ is in the range $y_0 - b \leq \varphi(x) \leq y_0 + b$ when $x_0 - h \leq x \leq x_0 + h$

with $h = \min\{a, \frac{b}{M}\}$ since the slope of the graph of the solution $y = \varphi(x)$ is at least $-M$ and at most M . If the graph of the solution $y = \varphi(x)$ crosses the lines $y = y_0 \pm b$, then the points of intersection with the abscissa are

$x_0 \pm b/M$. Therefore, the abscissa at the point where the integral curve goes out of the rectangle is less than or equal to $x_0 + b/M$ and is greater than or equal to $x_0 - b/M$.

To prove the theorem, we transform the initial value problem (2.1.11) into an integral equation. After integrating both sides of equation (2.1.11) from the initial point x_0 to an arbitrary value of x , we obtain

$$y(x) = y_0 + \int_{x_0}^x f(s, y(s)) ds. \quad (2.1.12)$$

Since the last equation contains an integral of the unknown function $y(x)$ it is called an integral equation. More precisely, the equation of the form (2.1.12) is called a Volterra integral equation of the second kind. This integral equation is equivalent to the initial value problem (2.1.11) in the sense that any solution of one is also a solution of the other.

We prove the existence and uniqueness of equation (2.1.12) using Picard's iteration method or the method of successive approximations. We start by choosing an initial function φ_0 , either arbitrarily or to approximate solution of equation (2.1.11) in some way. The simplest choice is

$$\varphi_0(x) = y_0.$$

Then this (constant) function satisfies the initial condition $y(x_0) = y_0$. The next approximation φ_1 is obtained by

substituting φ_0 for $y(s)$ in the right side of equation (2.1.12), namely,

$$\varphi_1(x) = y_0 + \int_{x_0}^x f(s, \varphi_0) ds = y_0 + \int_{x_0}^x f(s, y_0) ds.$$

Let us again substitute the first order approximation in the right-hand side of equation (2.1.12) to obtain

$$\varphi_2(x) = y_0 + \int_{x_0}^x f(s, \varphi_1(x)) ds.$$

Each successive substitution into equation (2.1.12) results in a sequence of functions. In general, if the n -th approximation $\varphi_n(s)$ has been obtained in this way, then the $(n+1)$ -th approximation is taken to be the result of substituting φ_n

in the right-hand side of equation (2.1.12). Therefore,

$$\varphi_{n+1}(x) = y_0 + \int_{x_0}^x f(s, \varphi_n(s)) ds \quad (2.1.13)$$

All terms of the sequence $\{\varphi_n(x)\}$ exist because

$$\left| \int_{x_0}^x f(s, \varphi(s)) ds \right| \leq \max |f(x, y)| \left| \int_{x_0}^x ds \right| = M |x - x_0| \leq b$$

for $|x - x_0| \leq b/M$, where $M = \max_{(x,y) \in \Omega} |f(x, y)|$.

The method of successive approximations gives a solution of equation (2.1.12) if and only if the successive approximations φ_{n+1} from equation (2.1.13) approach uniformly to a certain limit as $n \rightarrow \infty$. Then the sequence $\{\varphi_n(x)\}$ converges to a true solution $y = \varphi(x)$ as $n \rightarrow \infty$, which in fact is the unique solution of equation (2.1.12):

$$y = \varphi(x) = \lim_{n \rightarrow \infty} \varphi_n(x) \quad (2.1.14)$$

We can identify each element $\varphi_n(x)$ on the right-hand side of equation (2.1.14),

$$\varphi_n(x) = \varphi_0 + [\varphi_1(x) - \varphi_0] + [\varphi_2(x) - \varphi_1(x)] + \cdots + [\varphi_n(x) - \varphi_{n-1}(x)],$$

as the n -th partial sum of the telescopic series

$$\varphi(x) = \varphi_0 + \sum_{n=1}^{\infty} [\varphi_n(x) - \varphi_{n-1}(x)] \quad (2.1.15)$$

The convergence of the sequence $\{\varphi_n(x)\}$ is established by showing that the series (2.1.15) converges. To do this, we estimate the magnitude of the general term $|\varphi_n(x) - \varphi_{n-1}(x)|$.

We start with the first iteration:

$$|\varphi_1(x) - \varphi_0| = \left| \int_{x_0}^x f(s, y_0) ds \right| \leq M \int_{x_0}^x ds = M |x - x_0|.$$

For the second term we have

$$|\varphi_2(x) - \varphi_1(x)| \leq \int_{x_0}^x |f(s, \varphi_1(s)) - f(s, \varphi_0(s))| ds \leq L \int_{x_0}^x M(x - x_0) ds = \frac{ML(x - x_0)^2}{2}.$$

where L is the Lipschitz constant for the function $f(x, y)$, that is, $|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$. For the $n - th$ term,

we have

$$\begin{aligned} |\varphi_n(x) - \varphi_{n-1}(x)| &\leq \int_{x_0}^x |f(s, \varphi_{n-1}(s)) - f(s, \varphi_{n-2}(s))| ds \\ &\leq L \int_{x_0}^x |\varphi_{n-1}(s) - \varphi_{n-2}(s)| ds \leq L \int_{x_0}^x \frac{ML^{n-2}(s - x_0)^{n-1}}{(n-1)!} ds \\ &= \frac{ML^{n-1}|x - x_0|^n}{n!} \leq \frac{ML^{n-1}h^n}{n!}, \quad h = \max |x - x_0|. \end{aligned}$$

Substituting these results into the finite sum

$$\varphi_n(x) = \varphi_0 + \sum_{k=1}^n [\varphi_k(x) - \varphi_{k-1}(x)],$$

we obtain

$$\begin{aligned} |\varphi_n - \varphi_0| &\leq M|x - x_0| + \frac{ML|x - x_0|^2}{2} + \dots + \frac{ML^{n-1}|x - x_0|^n}{n!} \\ &= \frac{M}{L} \left[L|x - x_0| + \frac{L^2|x - x_0|^2}{2} + \dots + \frac{L^n|x - x_0|^n}{n!} \right] \end{aligned}$$

When n approaches infinity, the sum

$$L|x - x_0| + \frac{L^2|x - x_0|^2}{2} + \dots + \frac{L^n|x - x_0|^n}{n!}$$

approaches $e^{(L|x-x_0|)} - 1$. Thus, we have

$$|\varphi_n(x) - y_0| \leq \frac{M}{L} [e^{(L|x-x_0|)} - 1]$$

for any n . Therefore, by the Weierstrass M-test, the series (2.1.15) converges absolutely and uniformly on the interval

$|x - x_0| \leq h$. It follows that the limit function (2.1.14) of the sequence (2.1.13) is a continuous function on the interval

$|x - x_0| \leq h$. Sometimes a sequence of continuous functions converges to a limit function that is discontinuous, Next we will prove that $\varphi(x)$ is a solution of the initial value problem (2.1.11). First of all, $\varphi(x)$ satisfies the initial condition. In fact, from (2.1.13),

$$\varphi_n(x_0) = y_0, \quad n = 0, 1, 2, \dots$$

and taking limits of both sides as $n \rightarrow \infty$, we find $\varphi(x_0) = y_0$. Since $\varphi(x)$ is represented by a uniformly convergent series (2.1.15), it is a continuous function on the interval $x_0 - h \leq x \leq x_0 + h$.

Allowing n to approach ∞ on both sides of equation (2.1.13), we get

$$\varphi(x) = y_0 + \lim_{n \rightarrow \infty} \int_{x_0}^x f(s, \varphi_n(s)) ds. \quad (2.1.16)$$

Recall that the function $f(x, y)$ satisfies the Lipschitz condition for $|s - x_0| \leq h$:

$$|f(s, \varphi_n(s)) - f(s, \varphi(s))| \leq L|\varphi_n(s) - \varphi(s)|.$$

Since the sequence $\varphi_n(s)$ converges uniformly to $\varphi(s)$ on the interval $|s - x_0| \leq h$, it follows that the sequence $f(s, \varphi_n(s))$ also converges uniformly to $f(s, \varphi(s))$ on this interval. Therefore, we can interchange integration with

the limiting operation on the right-hand side of equation (2.1.16) to obtain

$$\varphi(x) = y_0 + \int_{x_0}^x \lim_{n \rightarrow \infty} f(s, \varphi_n(s)) ds.$$

Thus, the limit function $\varphi(x)$ is a solution of the integral equation (2.1.12) and, consequently, a solution of the initial value problem (2.1.11). In general, taking the limit under the sign of integration is not permissible, However, it is true for a uniform convergent sequence. Differentiating both sides of the last equality with respect to x and noting that the right-hand side is a differentiable function of the upper limit, we find

$$\varphi'(x) = f(x, \varphi(x)).$$

This completes the proof that the limit function $\varphi(x)$ is a solution of the initial value problem (2.1.11).

Uniqueness. Finally we will prove that $\varphi(x)$ is the only solution of the initial value problem (2.1.11). To start,

we assume the existence of another solution $y = \psi(x)$. Then

$$\varphi(x) - \psi(x) = \int_{x_0}^x [f(s, \varphi(s)) - f(s, \psi(s))] ds$$

for $|x - x_0| \leq h$. Setting $U(x) = |\varphi(x) - \psi(x)|$, we have

$$U(x) \leq \int_{x_0}^x |f(s, \varphi(s)) - f(s, \psi(s))| ds \leq L \int_{x_0}^x U(s) ds$$

By differentiating both sides with respect to x , we obtain $U'(x) - LU(x) \leq 0$. Multiplying by the integrating factor e^{-Lx} reduces the latter inequality to the following one:

$$[e^{-Lx}U(x)]' \leq 0$$

The function $e^{-Lx}U(x)$ has a nonpositive derivative and therefore does not increase with x . After integrating from x_0 to $(x > x_0)$, we obtain

$$e^{-Lx}U(x) \leq 0$$

since $U(0) = 0$. The absolute value of any number is positive; hence, $U(x) > 0$. Thus, $U(x) = 0$ for $x > x_0$. The case $x < x_0$ can be treated in a similar way. Therefore $\varphi(x) = \psi(x)$

2.2 Numerical Methods of IVP for a first-order:

Consider the differential equation with IVP(Initial Value Problem)

$$\begin{cases} y'(x) = f(x, y); \\ y(x_0) = y_0 \end{cases} \quad (2.2.1)$$

over the interval, $x \in [a; b]$.

We divide the interval $a \leq x \leq b$ into N segments of constant length h , called the step size. Thus the step size is $h = \frac{b-a}{N}$. This defines a set of equally spaced discrete points $a = x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n = b$, where $x_{i+1} = x_i + h$, $i = 1, 2, \dots, N$

Taylor Series Method of order Two (TS(2)).

we know

$$y(x) = y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + R_2(x)$$

has remainder term $R_2(x) = O(h^3)$. Setting $x = x_n$

we obtain

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{1}{2!}h^2y''(x_n) + O(h^3)$$

Neglecting the remainder term on the grounds that it is small leads to the formula

$$y_{n+1} = y_n + hy'_n + \frac{1}{2!}h^2y''_n \quad (2.2.2)$$

in which y_n, y'_n , and y''_n denote approximations to $y(x_n), y'(x_n)$ and $y''(x_n)$ respectively.

We shall refer to this as the TS(2) method, the value of y'_n can be computed from the IVP (2.2.1)

$$y'_n = f(x_n, y_n)$$

for y''_n we need to differentiate both sides of the ODE, as illustrated on the following formula

Apply the TS(2) method (2.2.2) to solve the IVP

$$\begin{cases} y'(x) = p(x)y(x) + q(x) \\ y(x_0) = y_0 \end{cases}$$

with $p(x)$ and $q(x)$ is functions continuous

In order to apply the formula (2.2.2) we must express $y''(x)$ in terms of $y(x)$ and x (it could also involve $y_0(x)$, but this can be substituted for from the ODE) with the chain rule we find

$$\begin{aligned} y''(x) &= \frac{d}{dx} [p(x)y(x) + q(x)] \\ &= p'(x)y(x) + p(x)y'(x) + q'(x) \\ &= p'(x)y(x) + p(x)[p(x)y(x) + q(x)] + q'(x) \\ &= [p'(x) + p(x)^2]y(x) + p(x)q(x) + q'(x) \end{aligned}$$

and so the TS(2) method is given

$$y_{n+1} = y_n + h[p(x)y(x) + q(x)] + \frac{1}{2!}h^2 \left[(p'(x) + p(x))y(x) + p(x)q(x) + q'(x) \right], n = 0, 1, \dots$$

Error of the Taylor series method is simply the first term of the discarded part of the series at some point $\xi \in (x_0, x_0 + h)$; i.e. if first n -term have been retained in obtaining the value of y then

Error of Taylor series will be $\leq \frac{h^n}{n!}y^{(n)}(\zeta), \zeta \in (x_0, x_0 + h)$ i.e $O(h^n)$

solve the initial value problem

$$y' = -4y + x^2; \quad y(0) = 1$$

from $x = 0$ to 1 with the Taylor series method of order two using $N = 10$, where the analytical solution of the IVP is

$$y(x) = \frac{31}{32} \exp(-4x) + \frac{1}{4}x^2 - \frac{1}{8}x + \frac{1}{32}$$

Taylor Method IVP for a first-order

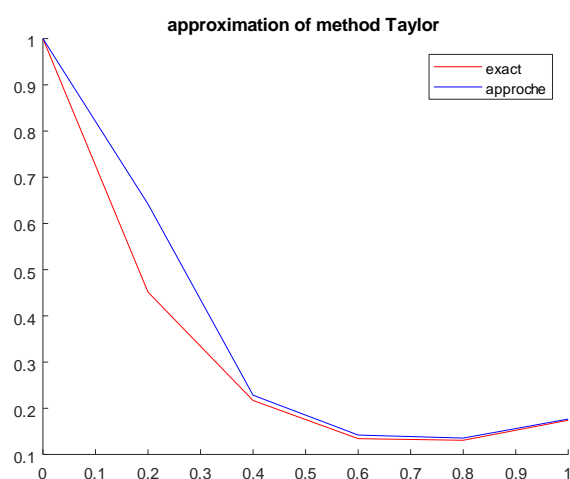
Solution The Taylor method

$$\begin{aligned} y(x+h) &= y(x) + hy'(x) + \frac{1}{2}h^2y''(x) + \frac{1}{3!}h^3y'''(\omega); \quad x < \omega < x+h \\ y(0) &= 1 \\ y'(x) &= -4y + x^2 \\ y''(x) &= -4y' + 2x \\ &= -4(-4y + x^2) + 2x \end{aligned}$$

which is computed

valuesof x	Exactsolution	Approxsolution	Error
0.000e+000	1.000000e+000	1.000000e+000	0.000000e+000
2.000e-001	4.515374e-001	4.642000e-001	1.266257e-002
4.000e-001	2.168373e-001	2.283821e-001	1.154483e-002
6.000e-001	1.341330e-001	1.420279e-001	7.894857e-003
8.000e-001	1.307384e-001	1.355377e-001	4.799304e-003
1.000e+000	1.739933e-001	1.767286e-001	2.735352e-003

Table of approximation by Taylor the.first order



The Taylor series method of first order

2.3 Numerical Methods of IVP for second order

Given a differential equation of the second order

$$y'' = f(x, y, y') \quad (2.3.1)$$

An initial value problem (abbreviated IVP) for a second-order differential equation is the problem of finding a solution $y(x)$ to equation (2.3.1) that satisfies an initial conditions $y(x_0)$ and $y'_0(x_0)$; where x_0 is some fixed value

of point and $y(x_0)$ and $y_0(x_0)$ are a fixed states. We write the IVP concisely as

$$IVP \quad \begin{cases} y'' = f(x, y, y'). \\ y(x_0) = y_0; \quad y'(x_0) = y_1 \end{cases} \quad (2.3.2)$$

Suppose we want to solve the following initial value problem on the interval $a \leq x \leq b$

$$y'' = f(x, y, y'), \quad y(x_0) = y_0; \quad y'(x_0) = y_1$$

or again

$$y'' + p(x)y' + q(x)y = f(x) :$$

If the functions p ; q and g are continuous on the interval $I : a \leq x \leq b$ containing the point $x = x_0$: Then there exists a unique solution $y(x)$ of the problem, and that this solution exists throughout the interval I : Conversely, neither existence nor uniqueness of a solution is guaranteed at a discontinuity of $p(t)$; $q(t)$; or $g(t)$.

Rather than seek a continuous solution defined at each point x ; we develop a strategy of discretization the problem to determine an approximation at discrete points in the interval of interest. Therefore, the plan is to replace the continuous function model (2.3.2) with an approximate discrete function model that is amenable to computer solution. We divide the interval $a \leq x \leq b$ into N segments of constant length h called the step size. Thus the step size is $h = \frac{b-a}{N}$. This defines a set of equally spaced discrete points $a = x_0 \leq x_1 \leq x_2 \leq x_3 \dots, x_N = b$, where $x_{i+1} = x_i + h$; $i = 0; 1; 2 \dots N$, Now, suppose we know the solution $y(x_i)$ and the first derivative $y'(x_i)$ of the initial value problem at the point x_i :

$$\begin{aligned} Y(x_0) &= \begin{pmatrix} y(x_0) \\ z(x_0) \end{pmatrix} = \begin{pmatrix} y(x_0) \\ y'(x_0) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ Y'(x) &= \begin{pmatrix} y'(x) \\ z'(x) \end{pmatrix} = \begin{pmatrix} y'(x) \\ y''(x) \end{pmatrix} = \begin{pmatrix} y'(x) \\ f(x, y, z) \end{pmatrix} = \begin{pmatrix} z(x) \\ z'(x) \end{pmatrix} \\ Y''(x) &= \begin{pmatrix} y''(x) \\ z''(x) \end{pmatrix} = \begin{pmatrix} z'(x) \\ z''(x) \end{pmatrix} = \begin{pmatrix} f(x, y, z) \\ z''(x) \end{pmatrix} \end{aligned}$$

Example 2.3.1 solve the intial value problem

$$y'' = -4xy' - (3 + 4x^2)y; \quad y(0) = 0, y'(0) = 1, \quad (8)$$

for $0 \leq x \leq 1$ by the séries of Taylor method order two with $N = 10$, the exact solution of initial problem is Known by

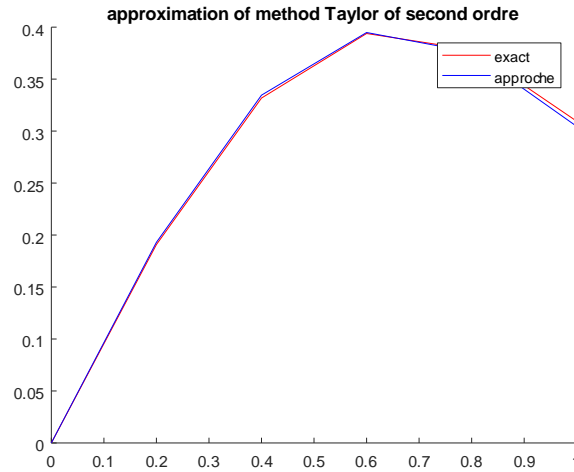
$$y(x) = \sin(x) \exp(-x^2).$$

$$\begin{aligned} Y(x) &= \begin{pmatrix} y(x) \\ z(x) \end{pmatrix} = \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ Y'(x) &= \begin{pmatrix} y'(x) \\ z'(x) \end{pmatrix} = \begin{pmatrix} z(x) \\ -4xz - (3 + 4x^2)y \end{pmatrix} \\ Y''(x) &= \begin{pmatrix} z'(x) \\ -4z - 4xz' - 8xy - (3 + 4x^2)z \end{pmatrix} \\ &= \begin{pmatrix} -4xz - (3 + 4x^2)y \\ 4xz' - (7 + 4x^2)z - 8xy \end{pmatrix} \end{aligned}$$

which is computed

values of x	Exact solution	Approx solution	Error
0.000e+000	0.000000e+000	0.000000e+000	0.000000e+000
2.000e-001	1.908794e-001	1.930500e-001	2.170605e-003
4.000e-001	3.318404e-001	3.346036e-001	2.763185e-003
6.000e-001	3.939377e-001	3.948634e-001	9.256742e-004
8.000e-001	3.782564e-001	3.759754e-001	2.281027e-003
1.000e+000	3.095599e-001	3.046570e-001	4.902875e-003

Table of aproximation by Taylor method of second Order



2.4 Transformation of the initial value problem into Volterra integral equation second kind

First order

$$\begin{aligned}
 y' &= f(x, y), & y(x_0) &= y_0 \\
 \int_{x_0}^x y'(x) dx &= \int_{x_0}^x f(x, y) dx \\
 y(x) &= y_0 + \int_{x_0}^x f(x, y) dx
 \end{aligned}$$

Second order

$$\begin{aligned}
 y' &= f(x, y, y'), & y(x_0) &= y_0; \quad y'(x_0) = y_1, \\
 \int_{x_0}^x y''(x) dx &= \int_{x_0}^x f(x, y, y') dx, \\
 y'(x) &= y_1 + \int_{x_0}^x f(x, y, y') dx \\
 \int_{x_0}^x y'(x) dx &= y_1(x - x_0) + \int_{x_0}^x \int_{x_0}^x f(z, y, y') dz dx \\
 y(x) &= y_0 + y_1(x - x_0) + \int_{x_0}^x (x - z) f(z, y, y') dz
 \end{aligned}$$

Chapter 3

Numerical Solution of Volterra Integral Equations

A typical form of an integral equation in $u(x)$ is at the form

$$u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x, t)u(t)dt \quad (3.0.1)$$

Where $K(x, t)$ is called the kernel of the integral (3.0.1) and $\alpha(x)$ and $\beta(x)$ are the limits of integration. It can be easily observed that the unknown function $u(x)$ appears under the integral sign. It is to be note here that both the kernel $K(x, t)$ and the function $f(x)$ in (3.0.1) are given functions and λ is a constant parameter.

The most standard form of Volterra integral equations is of the form

$$u(x)h(x) = f(x) + \lambda \int_a^x K(x, t)u(t)dt \quad (3.0.2)$$

where the limits of integration are function of and the unknown function $u(x)$ appears linearly under the integral sign. if the function $h(x) = 1$, then the (3.0.2) simply becomes

$$u(x) = f(x) + \lambda \int_a^x K(x, t)u(t)dt \quad (3.0.3)$$

and this equation is known as Volterra integral equations of the second kind; whereas if $h(x) = 0$, the equation (3.0.1) becomes

$$f(x) + \lambda \int_a^x K(x, t)u(t)dt = 0 \quad (3.0.4)$$

Which is known as Volterra integral equations is of the first kind. In this paper we introduce a modified method which is based on the Simpson's quadrature formula. The idea is to approximate the solution of the above equation in even number of equally spaced points (or a given mesh). Then in the subinterval $[s, s + 2h]$ we have

$$\begin{aligned} \int_s^{s+2h} k(x, t) \varphi(t) dt &= \frac{h}{3} (h(x, s) \varphi(s) + 4k(x, s + h) \varphi(s + h) + k(x, s + 2h) \varphi(s + 2h)) \\ &\quad - \frac{h^5}{90} (k(x, \zeta) \varphi(\zeta))^{(4)} \end{aligned}$$

This indicates that the error $E(h)$ of integration over two segments by Simpson's rule is proportional to h^5 . Also, we note that if the segment width h is halved to $\frac{h}{2}$, then $E(\frac{h}{2}) \approx -2 \frac{(h/2)^5}{90} (k(x, \zeta) \varphi(\zeta))^{(4)}$.

3.1 Development of the Method

consider a Volterra integral equation given by

$$\varphi(x) = g(x) + \int_a^x k(x, t) \varphi(t) dt, \quad a \leq x \leq b \quad (3.1.1)$$

Let the interval $[a, b]$ be finite and partitioned by $2n$ equally spaced points (subdivision of smaller step h)

$$x_0 = a < x_1 < \dots < x_{2j-1} < x_{2j} < \dots < x_{2n}.$$

We solve the problem by marching in time. The approximation of (3.1.1) in the even nodes (x_{2j}) is given by

$$\begin{aligned} \varphi(x_{2j}) &= g(x_{2j}) + \int_a^{x_{2j}} k(x_{2j}, t) \varphi(t) dt \\ &= g(x_{2j}) + \sum_{i=0}^{j-1} \int_a^{t_{2i+2}} k(x_{2j}, t) \varphi(t) dt \end{aligned}$$

which can be rewritten as

$$\varphi_{2j} = g(x_{2j}) + \sum_{i=0}^{j-1} \int_a^{t_{2i+2}} k(x_{2j}, t) \varphi(t) dt.$$

Using Simpson's quadratic formula, the above discrete equation becomes

$$\varphi_{2j} = g(x_{2j}) + \sum_{i=0}^{j-1} \frac{h}{3} (k_{2j,2i} \varphi_{2i} + 4k_{2j,2i+1} \varphi_{2i+1} + k_{2j,2i+2} \varphi_{2i+2}).$$

For a smaller step h , an approximation to φ_{2j} can then be computed by replacing φ_{2i+1} by the average $\frac{\varphi_{2i} + \varphi_{2i+2}}{2}$,

$$\begin{aligned}
 \varphi_{2j} &= g_{2j} + \sum_{i=0}^{j-1} \frac{h}{3} \left[k_{2j,2i} \varphi_{2i} + 4k_{2j,2i} \frac{\varphi_{2i} + \varphi_{2i+2}}{2} + k_{2j,2i} \varphi_{2i+2} \right] \\
 &= g_{2j} + \sum_{i=0}^{j-1} \frac{h}{3} [(k_{2j,2i} + 2k_{2j,2i+1}) \varphi_{2i} + (2k_{2j,2i+1} + k_{2j,2i+2}) \varphi_{2i+2}] \\
 &= g_{2j} + \sum_{i=0}^{j-1} \frac{h}{3} (k_{2j,2i} + 2k_{2j,2i+1}) \varphi_{2i} + \sum_{i=0}^{j-1} (2k_{2j,2i+1} + k_{2j,2i+2}) \varphi_{2i+2} \\
 &= g_{2j} + \sum_{i=0}^{j-1} \frac{h}{3} (k_{2j,2i} + 2k_{2j,2i+1}) \varphi_{2i} + \sum_{i=1}^j \frac{h}{3} (2k_{2j,2i-1} + k_{2j,2i}) \varphi_{2i}, \\
 \varphi_{2j} &= g_{2j} + \frac{h}{3} (k_{2j,0} + 2k_{2j,1}) \varphi_0 + \frac{h}{3} (2k_{2j,2j-1} + k_{2j,2j}) \varphi_{2j} + \frac{2h}{3} \sum_{i=1}^{j-1} (k_{2j,2i-1} + k_{2j,2i} + k_{2j,2i+1}) \varphi_{2i}
 \end{aligned}$$

In general, for $j = 1, 2, \dots, n$

$$\varphi_{2j} \left(1 - \frac{h}{3} (2k_{2j,2j-1} + k_{2j,2j}) \right) = g_{2j} + \frac{h}{3} (k_{2j,0} + 2k_{2j,1}) \varphi_0 + \frac{2h}{3} \sum_{i=1}^{j-1} (k_{2j,2i-1} + k_{2j,2i} + k_{2j,2i+1}) \varphi_{2i}.$$

clearly from (3.1.1) the value of $\varphi(0)$ is $g(0)$, so $\varphi_0 = g_0$.

3.1.1 Numerical Experiments

Volterra integral equation IVP for a first-order

Example 3.1.1 solve the initial value problem

$$y' = -4y + x^2; \quad y(0) = 1$$

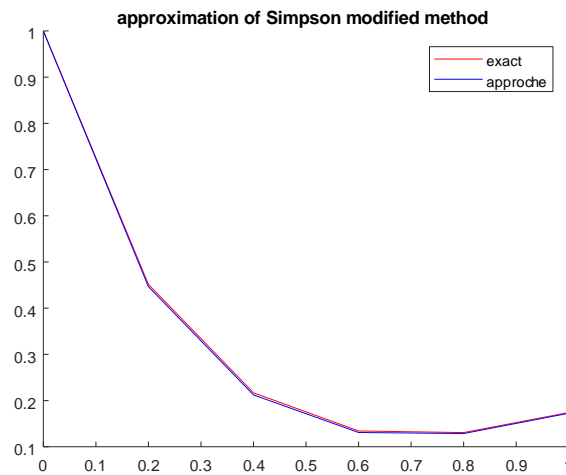
On the other hand, the transformation of the initial value problem into Volterra integral equation gives the best approximation solution using the modified Simpson method

$$\begin{aligned}
 \int_0^x y'(x) dx &= \int_0^x -4y(x) dx + \int_0^x x^2 dx; \\
 y(x) - y(0) &= \int_0^x -4y(x) dx + \frac{1}{3} x^3 \\
 y(x) &= \int_0^x -4y(x) dx + \frac{1}{3} x^3 + 1
 \end{aligned}$$

wich is computed

valuesof x	Exactsolution	Approxsolution	Error
0.000e+000	1.000000e+000	1.000000e+000	0.000000e+000
2.000e-001	4.515374e-001	4.465741e-001	4.963360e-003
4.000e-001	2.168373e-001	2.122737e-001	4.563589e-003
6.000e-001	1.341330e-001	1.309179e-001	3.215093e-003
8.000e-001	1.307384e-001	1.286487e-001	2.089678e-003
1.000e+000	1.739933e-001	1.726402e-001	1.353109e-003

Table of aproximation by modified Simpson method



The solution with the modified Simpson method

Example 3.1.2 solve the intial problem folowing

$$y'' = -4xy' - (3 + 4x^2)y; \quad y(0) = 0, y'(0) = 1, \quad (8)$$

for $0 \leq x \leq 1$ by the séries of Taylor method order two with $N = 10$, the exact solution of initial problem is Known by

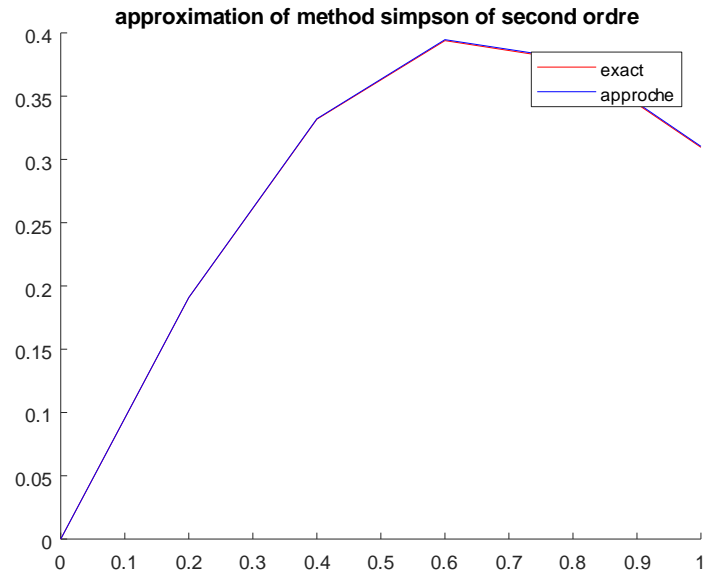
$$y(x) = \sin(x) \exp(-x^2).$$

integral equation

$$\begin{aligned}
\int_0^x y''(x) dx &= \int_0^x -4xy' dx - \int_0^x (3 + 4x^2)y dx \\
y'(x) - y'(0) &= -4xy(x) + \int_0^x -4xy dx - \int_0^x (3 + 4x^2)y dx; \\
y'(x) &= \int_0^x (1 - 4x^2)y(x) dx - 4xy(x) + 1 \\
\int_0^x y'(x) dx &= \int_0^x \int_0^x (1 - 4x^2)y(x) dx - \int_0^x 4xy(x) dx + x \\
y(x) - y(0) &= \int_0^x ((1 - 4z^2)(x - z) - 4z)y(z) dz + x \\
y(x) &= \int_0^x ((1 - 4z^2)(x - z) - 4z)y(z) dz + x
\end{aligned}$$

values of x	Exact solution	Approx solution	Error
0.000e+000	0.000000e+000	0.000000e+000	0.000000e+000
2.000e-001	1.908794e-001	1.909270e-001	4.764010e-005
4.000e-001	3.318404e-001	3.321603e-001	3.198637e-004
6.000e-001	3.939377e-001	3.946794e-001	7.417302e-004
8.000e-001	3.782564e-001	3.792457e-001	9.892349e-004
1.000e+000	3.095599e-001	3.103804e-001	8.205444e-004

Table of approximation by modified Simpson method



The modified Simpson method $y'' = -4xy' - (3 + 4x^2)y$; $y(0) = 0, y'(0) = 1$

transformed into VIE $y(x) - \int_0^x ((1 - 4z^2)(x - z) - 4z)y(z) dz = x$

In this section we describe some of the numerical experiments performed in solving the linear Volterra integral equations. In all cases we chose the right-hand side $g(x)$ in such a way that we know the exact solution. This exact solution is used only to show that the numerical solution obtained with our method is correct. Then, in such example, we calculate the absolute errors at some points (even points for example)

Conclution

In my work we studied the compare between the solution differential equation and the Integral solution, For this we remark that all initial value problem can be transformed into Volterra integral equation of the two first kind, where the numerical solution of those equations using the modified Simpson method is better than the Taylor method, and also we conclude that for sufficiently small h we get a good accuracy, since by reducing the step size length the least square error will be reduced. both the two methods.

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Abstract :

In this work, we solve initial value problem for a two first order numerically using Taylor series. On the other hand, we show that after the transformation of the (IVP) into Volterra integral equation of the second kind, its numerical solution using modified Simpson method are better than Taylor method.

Résumé:

Dans ce travail, on résout le problème de valeur initial du premier et deuxième ordre numériquement utilisant la méthode de Taylor. Puis, on transforme (PVI) à une équation intégrale de Volterra , sa solution numérique obtenu par la méthode de Simpson modifié. est la meilleure par rapport à celle de Taylor.

ملخص:

في هذا العمل، قمنا بحل المعادلة التفاضلية الدرجة الأولى والثانية ذات القيم الابتدائية باستخدام طريقة تايلور. ثم نقوم بتحويل المعادلة التفاضلية إلى معادلات تكاملية من نوع فولتيرا لنقوم بحلها بطريقة سيمبسون المعدلة الدرجة الأولى والثانية لنجرب أن طريقة سيمبسون المعدلة أحسن من تايلور.